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Geršgorin type theorems for quaternionic matrices

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Dedicated to Roger Horn on the occasion of his 65th birthday with admiration of his great passion and contribution to linear algebra and matrix analysis

Abstract

This paper aims to set an account of the left eigenvalue problems for real quaternionic (finite) matrices. In particular, we will present the Geršgorin type theorems for the left (and right) eigenvalues of square quaternionic matrices. We shall conclude the paper with examples showing and summarizing some differences between complex matrices and quaternionic matrices and right and left eigenvalues of quaternionic matrices.

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1. Introduction

Quaternions were introduced by the Irish mathematician Sir William Rowan Hamilton (1805–1865) in 1843 as he looked for ways of extending complex numbers to higher spatial dimensions. Ever since his creation of quaternions, Hamilton spent the rest of his life obsessed with them and their applications [10]. Nevertheless he probably never thought that one day in the future

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the quaternions he had invented would be used in programming video games and controlling spacecrafts [15,21].

As an area of mathematics quaternions have been extensively studied. For a while, they were fashionable and taught as the only advanced mathematics in Dublin and some American universities [2]. Nowadays quaternions are not only part of contemporary mathematics (algebra, analysis, geometry, and computation; see, e.g., [7,9,14,15,18,23]), but they are also widely and heavily used in computer graphics, control theory, signal processing, altitude control, physics, and mechanics (mainly for representing rotations and orientations of objects in three-dimensional space). For example, spacecraft altitude-control systems are commanded in terms of quaternions. See [1,15] and the references therein.

The research on mathematical objects associated with quaternions has been active in recent years; there are many research papers published in a variety of journals each year and different approaches have been taken for different purposes. The present article is concerned with (finite) quaternionic matrices and particularly the left eigenvalue problem.

As expected, the main obstacle in the study of quaternionic matrices is the noncommutative multiplication of quaternions. A problem on quaternions may be viewed of interest if the result, solution or conclusion is rather different than that of the complex case or the method resolving the problem has to be novel. The theory on right eigenvalues of quaternionic matrices has been well established [8,16,17,26], while little is known for left eigenvalues. The left eigenvalue problem was raised in [5, p. 217]. In the quaternionic setting

$$Ax = \lambda x \quad \text{and} \quad Ax = x\lambda$$

are two very different systems of equations; so different that there is in general no connection between them. One of the properties of the left eigenvalues that stands out is that a quaternion λ is a left eigenvalue of a square matrix A if and only if $A - \lambda I$ is singular, since $Ax = \lambda x$ is equivalent to $(A - \lambda I)x = 0$, while this is not true for right eigenvalues: $Ax = x\lambda$ cannot be rewritten in the "nice" form. The investigation of left eigenvalues is mainly driven by purely mathematical interest. As for a solution to any math problem, we are concerned with "existence, uniqueness, and structure". It has been evident that topological approaches are effective ways for tackling the existence problems. Wood [25] shows that every square quaternionic matrix has at least one left eigenvalue, so does Baker [3] for right eigenvalues as a counterpart to Wood's proof. (We remark here that a footnote of [16] asserts that a left eigenvalue does not exist in general. This assertion is false.) We study the left eigenvalue problem from the linear algebra and matrix-theoretic points of view and make a comparison with the complex matrix theory. What remains valid or invalid for quaternionic matrices, particularly for left eigenvalue, is a major point of our study.

For linear algebraists and matrix theorists, some basic questions on quaternionic matrices are of interest and still need to be answered. For example,

Question 1: Is there an elementary proof (without using homotopy) for the existence of the left eigenvalues? (see [25] for a topological proof).

Question 2: If A is an $n \times n$ quaternionic matrix having finite, say k, distinct left eigenvalues, is it true that $k \leqslant n$? (see [13]).

Question 3: If the left and right spectra of a square matrix are both finite, do they coincide? (see [19]), as is shown for the 2×2 case [13].

We shall mainly adopt the notation and terminology in [26]. For convenience, recall that, as usual, $\mathbb{R}$ and $\mathbb{C}$ are the sets of real and complex numbers, respectively. We denote by $\mathbb{H}$ (for Hamilton) the set of real quaternions:

$$\mathbb{H} = \{a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

with

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

For $q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \in \mathbb{H}$, the conjugate of q is $q^* = \bar{q} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$ and the norm, length, or modulus of q is

$$|q| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}.$$

Two quaternions x and y are said to be *similar* if there exists a nonzero quaternion q such that $x = q^{-1}yq$. It is known (see, e.g., [26, Theorem 2.2]) that $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ is similar to $y = y_0 + y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$ if and only if $x_0 = y_0$ and $x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2$. For instance, $\mathbf{i}$ and $\mathbf{j}$ are similar.

Let $\mathbb{F}^n$ and $\mathbb{F}^{m\times n}$ be respectively the collections of all n-column vectors and $m \times n$ matrices with entries in $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. For $v \in \mathbb{F}^n$, v^{t} is the transpose of v. If $v = [v_1, \ldots, v_n]^{\mathrm{t}}$, then $\bar{v} = [\bar{v}_1, \ldots, \bar{v}_n]^{\mathrm{t}}$ is the conjugate of v and $v^* = [\bar{v}_1, \ldots, \bar{v}_n]$ is the conjugate transpose of v. The norm of v is defined to be $\|v\| = \sqrt{v^*v}$. For an $m \times n$ matrix $A = (a_{ij})$ with quaternionic entries, the conjugate transpose of A is the $n \times m$ matrix $A^* = \overline{A}^{\mathrm{t}} = (a_{ji}^*)$.

Let $A \in \mathbb{H}^{n\times n}$. A quaternion λ is said to be a *right* eigenvalue of A if $Ax = x\lambda$ for some nonzero quaternionic (column) vector x. Similarly λ is a *left* eigenvalue if $Ax = \lambda x$. (It is called a *singular eigenvalue* in [6, p. 370].) Some existing examples show that a quaternionic matrix may have finite or infinitely many left eigenvalues. One of the difficulties in studying the left eigenvalues is that the left eigenvalues are not unitarily invariant; to be precise, a matrix A and U^*AU may have different left eigenvalues, where U is a unitary matrix. Another difficulty, from the point of view of operators, is that $A : x \mapsto Ax$ regarded as an action on the n-column vectors of quaternions is linear on the *right* Hilbert space (or module) $\mathbb{H}^n$ but nonlinear on the *left* Hilbert space $\mathbb{H}^n$.

We denote the left and right spectra of a square quaternionic matrix A by $\sigma_{\mathrm{l}}(A)$ and $\sigma_{\mathrm{r}}(A)$, respectively. It is known that they are never empty and that when A is real, they coincide [26, Theorem 5.2]. Moreover, if A is a complex matrix and λ is a complex eigenvalue of A, then $\lambda \in \sigma_{\mathrm{l}}(A) \cap \sigma_{\mathrm{r}}(A)$.

The right spectrum $\sigma_{\mathrm{r}}(A)$ can be obtained by computing the (ordinary) eigenvalues of the complex representative (or adjoint) χ_A of the matrix A. Here

$$\chi_A = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} \quad \text{if } A = A_1 + A_2\mathbf{j},$$

where A_1 and A_2 are complex matrices. We note that every quaternionic matrix A can be uniquely written as $A = A_1 + A_2\mathbf{j}$ with A_1 and A_2 complex. However

$$\overline{A} \neq \overline{A}_1 + \overline{A}_2\mathbf{j}, \quad A^* \neq A_1^* + A_2^*\mathbf{j}, \quad \text{but} \quad A^{\mathrm{t}} = A_1^{\mathrm{t}} + A_2^{\mathrm{t}}\mathbf{j}.$$

The right eigenvalue theory of quaternionic matrices parallels that of the complex eigenvalues of complex matrices in some sense; the right spectrum $\sigma_{\mathrm{r}}(A)$ consists of the regular complex eigenvalues of the complex matrix χ_A and the elements in their quaternionic conjugacy equivalence classes [16,26]. In addition, the classical matrix decompositions such as SVD and QR decompositions associated with the right eigenvalues all can be carried over to the quaternionic matrices. In contrast, the behavior of left eigenvalues is quite unexpected. Some well-known facts in complex matrix theory may no longer be valid for left eigenvalues. For instance, a positive definite matrix (in the usual sense $x^*Ax > 0$ for all nonzero x) may have negative left eigenvalues; a matrix with distinct left eigenvalues may not be diagonalizable; there are no decompositions in terms of left eigenvalues, etc. Nevertheless, the left eigenvalues are not totally "out of control". In fact they are all bounded, as the Geršgorin type theorems for quaternionic matrices reveal in the next section.

2. Geršgorin theorem for quaternionic matrices

This section is devoted to the localization of the left eigenvalues. Now that the left eigenvalues exist, how do we find them? Although the left eigenvalue problems are difficult to deal with in general, they do share some properties with complex matrices. In fact they sometimes even behave "better" than the right eigenvalues. We shall see that left eigenvalues abide by some rules of complex matrices but the right eigenvalues may not. Even though we cannot characterize the left eigenvalues in "exact" form in general, we hope to locate them as closely as possible. This section aims to do so to some extent.

We begin with a result of Huang and So [13, Theorem 2.3].

Lemma 1 [13]. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a* 2×2 *quaternionic matrix. Then*

$$\sigma_l(A) = \{a, d\} \quad if\ bc = 0,$$

otherwise

$$\sigma_l(A) = \{a + bl : \ l^2 + b^{-1}(a - d)l - b^{-1}c = 0\}. \tag{1}$$

With this lemma we obtain a necessary condition for a quaternion to be a left eigenvalue of a 2×2 quaternionic matrix.

Theorem 1. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a* 2×2 *quaternionic matrix. If* λ *is a left eigenvalue of* A, *then* (*but not conversely*)

$$|\lambda - a||\lambda - d| = |b||c|. \tag{2}$$

Proof. If $bc = 0$, then $\lambda = a$ or d, (2) is obvious. Otherwise, by Lemma 1, writing $\lambda = a + bl$, we have

$$|\lambda - a||\lambda - d| = |bl||a + bl - d| = |b||bl^2 + (a - d)l| = |b||c|. \qquad \square$$

Remark. As is well known, for the complex matrix case, if $\lambda \in \mathbb{C}$ is an eigenvalue of a 2×2 complex matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\lambda I - A) = 0$. Thus

$$(\lambda - a)(\lambda - d) = bc, \tag{3}$$

which implies (not conversely)

$$|\lambda - a||\lambda - d| = |b||c|. \tag{4}$$

For the quaternionic case, the determinant makes no sense at this point. So the idea of using the determinant no longer works. In fact, for some right eigenvalues, neither (3) nor (4) holds. This is seen as one checks with $A = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{bmatrix}$ and $-\mathbf{i} \in \sigma_r(A)$. In contrast and interestingly, for left eigenvalues (4) holds, as we have shown in Theorem 1, but (3) does not, as we take $A = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{j} & 0 \end{bmatrix}$. Then $\sigma_{\mathrm{l}}(A) = \{\pm\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})\}$. It follows that (3) does not hold for any $\lambda \in \sigma_{\mathrm{l}}(A)$.

Our next observation, as in the complex case, is an upper bound for all left and right eigenvalues. It asserts that the left and right spectra are bounded. The proof is essentially the same as that of the complex case, thus omitted.

Theorem 2. *Let $A = (a_{ij})$ be an $n \times n$ quaternionic matrix and let $\lambda \in \mathbb{H}$ be a left or right eigenvalue of A. Then*

$$|\lambda| \leqslant \max_i \sum_{j=1}^{n} |a_{ij}| := R.$$

The result is not surprising, however it is significant in view that not much is known for left eigenvalues. The theorem says that all of the left as well as right eigenvalues, no matter how many, lie in the R-neighborhood of the origin. On the other hand, if A is row diagonally dominant, then all the left and right eigenvalues will be away from the origin:

$$|\lambda| \geqslant \min_i\{|a_{ii}| - R_i(A)\},$$

where

$$R_i(A) = \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad 1 \leqslant i \leqslant n,$$

and the matrix is nonsingular. Thus A is singular (not invertible) if and only if 0 is an eigenvalue (left or right) and if A is strictly row or column diagonally dominant, then A is invertible. This coincides with what we have in the complex case and sheds some light on the localization of the left eigenvalues, and we shall pursue further in this direction.

Let $A \in \mathbb{H}^{n\times n}$ and define the left and right spectral radii of A to be

$$\rho_{\mathrm{l}}(A) = \max\{|l| : l \in \sigma_{\mathrm{l}}(A)\} \quad \text{and} \quad \rho_{\mathrm{r}}(A) = \max\{|\lambda| : \lambda \in \sigma_{\mathrm{r}}(A)\}.$$

As shown in the later Example 3.11, $\rho_{\mathrm{l}}(A)$ and $\rho_{\mathrm{r}}(A)$ are not the same for

$$A = \begin{bmatrix} \mathbf{j} & \mathbf{i} \\ \mathbf{i} & -\mathbf{j} \end{bmatrix},$$

since

$$\sigma_{\mathrm{l}}(A) = \{0\}, \qquad \sigma_{\mathrm{r}}(A) = \{2q^{-1}\mathbf{i}q : 0 \neq q \in \mathbb{H}\}.$$

Thus

$$\rho_{\mathrm{l}}(A) = 0 \quad \text{and} \quad \rho_{\mathrm{r}}(A) = 2.$$

Theorem 3. *Let $A = (a_{ij})$ be an $n \times n$ quaternionic matrix. Then*

$$\rho_{\mathrm{l}}(A), \rho_{\mathrm{r}}(A) \leqslant \max_{\|x\|=1} \|Ax\|.$$

Proof. Let λ be a left or right eigenvalue of A with eigenvector v. Then

$$v^*A^*Av = |\lambda|^2 v^*v \quad \text{or} \quad \|Av\| = |\lambda|\|v\|.$$

The inequalities then follow immediately. $\square$

Note that the right spectral radius of a quaternionic matrix A can be computed through the complex representative since $\rho_{\mathrm{r}}(A) = \rho(\chi_A)$. Although we do not yet have a way to compute the left spectral radius in general, this theorem gives us some hope of further estimating quaternionic eigenvalues by vector or matrix norms.

The famous Geršgorin theorem is one of the fundamental theorems in complex matrix theory. It ensures that all the eigenvalues of a matrix are contained in the Geršgorin discs. A Geršgorin disc of A is a set in the complex plane

$$\{z \in \mathbb{C} : \; |z - a_{ii}| \leqslant R_i(A)\}.$$

The Geršgorin theorem for the complex matrix case states that every eigenvalue is contained in some disc and if k of those discs are connected, then this connected region contains exactly k eigenvalues (see, e.g., [11, p. 344]).

Our purpose of this section is to look into this theorem for quaternionic matrices. All terms are similarly defined as in the complex case and we call

$$\{z \in \mathbb{H} : \; |z - a_{ii}| \leqslant R_i(A)\}$$

a *Geršgorin ball* (in parallel to Geršgorin disc in the plane) or simply a *ball*.

A Geršgorin ball is a subset of the quaternions (four-dimensional space over $\mathbb{R}$). It may not contain any complex numbers. For instance, $\{q \in \mathbb{H} : \; |q - \mathbf{j}| < \frac{1}{2}\}$ does not touch the complex plane. In general, a Geršgorin ball $\{q \in \mathbb{H} : \; |q - a| < r\}$ contains a complex number if and only if $r > \sqrt{a_2^2 + a_3^2}$, where $a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $a_0, a_1, a_2, a_3 \in \mathbb{R}$.

First of all, a quaternionic matrix may have infinitely many (left or right) eigenvalues. So the k-eigenvalue part of the above statement does not hold in general for quaternionic matrices. However something similar can be said.

We first inspect the 2×2 case, showing that if a 2×2 matrix has infinitely many left eigenvalues, then the two balls must be connected, and that if the matrix has finite (2 different) left eigenvalues, then the two balls are disjoint and each ball contains a left eigenvalue.

Theorem 4. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a* 2×2 *quaternionic matrix. If* A *has infinitely many* (*distinct*) *left eigenvalues*, *then the two Geršgorin balls intersect.*

Proof. All we need to show is $|a - d| < |b| + |c|$. Since A has infinitely many left eigenvalues, by Huang and So [13, Theorem 3.2], $b^{-1}(a - d)$ and $b^{-1}c$ are real numbers, and moreover $[b^{-1}(a - d)]^2 + 4b^{-1}c < 0$. It follows that

$$0 \leqslant [b^{-1}(a - d)]^2 < -4b^{-1}c,$$

which yields

$$|a - d|^2 < 4|b||c|.$$

Thus

$$|a - d| < 2\sqrt{|b||c|} \leqslant |b| + |c|. \qquad \square$$

The converse of the previous theorem is false; one may take $A = \begin{bmatrix} 1 & 3 \\ 0 & \mathbf{i} \end{bmatrix}$ as a counterexample. The next theorem for 2×2 quaternionic matrices is a restatement of the Geršgorin theorem for the 2×2 complex matrices. The proof however is quite different. For the complex case, a straightforward proof is available, here we have to use a continuity argument.

Theorem 5. *Let A be a 2×2 quaternionic matrix. If the two Geršgorin balls of A are disjoint, then A has two distinct left eigenvalues and each ball contains a left eigenvalue.*

Proof. According to the previous theorem, A must have finite left eigenvalues, and, by [13, Theorem 3.1], A has at most two distinct left eigenvalues.

To show that A has exactly two left eigenvalues and that each ball contains one, let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad A_\epsilon = \begin{bmatrix} a & \epsilon b \\ \epsilon c & d \end{bmatrix}, \quad 0 \leqslant \epsilon \leqslant 1.$$

There is not much to show if $b = 0$ or $c = 0$. Note that A_0 has two left eigenvalues a and d. By Lemma 1, l is a left eigenvalue of A_ϵ if and only if

$$l = a + \epsilon b \lambda$$

for some λ such that

$$\epsilon \lambda^2 + b^{-1}(a - d)\lambda - \epsilon b^{-1} c = 0.$$

The solution λ to the equation depends continuously on ϵ. Thus $l := l(A_\epsilon)$ is a continuous function of ϵ. (λ may be a multiple-valued function. Pick and fix one branch.) So $\{l(A_\epsilon) : 0 \leqslant \epsilon \leqslant 1\}$ is a continuous curve starting from a ending at a left eigenvalue of $A = A_1$. And this curve is entirely contained in the ball. Thus $l(A)$, a left eigenvalue of A, is contained in the ball. $\square$

Example 2.1. For matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$, A has two eigenvalues 5.3 and 1.7, one is close to 5 and the other is close to 2. For matrix $B = \begin{bmatrix} \mathbf{i} & \frac{1}{2} \\ \frac{1}{2} & \mathbf{j} \end{bmatrix}$, the theorem ensures that B has two distinct left eigenvalues.

We next generalize the above theorems to the square quaternionic matrices of higher dimensions. For this purpose, we need the continuity property of the left and right eigenvalues on the entries of the matrix.

Lemma 2. *The left (and right) eigenvalues of an n-square quaternionic matrix are continuous functions of the entries of the matrix.*

Proof. Let χ_Q be the complex representative of the quaternionic matrix Q. Then $\chi : Q \mapsto \chi_Q$ is isomorphic from $\mathbb{H}^{n\times n}$ to $\mathbb{C}^{2n\times 2n}$.

For the left eigenvalue case, l is a left eigenvalue of a quaternionic matrix A, i.e., $Ax = lx$ for some $x \neq 0$ if and only if $(A - lI_n)x = 0$ which holds if and only if $\chi_{(A-lI_n)}\chi_x = 0$. It follows that l is a left eigenvalue of A if and only if

$$\det[\chi_{(A-lI_n)}] = 0.$$

For the right eigenvalue case, it is known [26, Theorem 8.1(5)] that λ is a right eigenvalue if and only if

$$\det[\lambda I_{2n} - \chi_A] = 0.$$

These determinantal equations yield the continuity of the left and right eigenvalues as functions of the entries of the matrix. □

We are now ready to show the Geršgorin theorems for the left and right eigenvalues. The proof for the left one is the same as that in the complex case [11, p. 344], while the proof for the right is slightly different. We thus present the one for the right and omit the one for the left.

Theorem 6 (Geršgorin theorem for left eigenvalues). *Let $A = (a_{ij})$ be an $n \times n$ matrices of quaternions. Then all the left eigenvalues of A are located in the union of n Geršgorin balls $\{q \in \mathbb{H} : |q - a_{ii}| \leqslant R_i(A)\}$. That is,*

$$\sigma_l(A) \subseteq \bigcup_{i=1}^{n} \{q \in \mathbb{H} : |q - a_{ii}| \leqslant R_i(A)\}.$$

In most circumstances, right eigenvalues are more useful than the left ones and they "behave" better than the left ones, but not for this case.

Example 2.2. A right eigenvalue is not necessarily contained in a Geršgorin ball. Take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{bmatrix}.$$

Then A has only two left eigenvalues 1 and $\mathbf{i}$, while the right eigenvalues are 1 and $q^{-1}\mathbf{i}q$, $q \in \mathbb{H}$. In particular, $-\mathbf{i}$ is a right eigenvalue of A. However it is not contained in any Geršgorin ball.

Fortunately we can show that every connected region of Geršgorin balls will contain some right eigenvalues.

Theorem 7 (Geršgorin theorem for right eigenvalues). *Let $A = (a_{ij})$ be an $n \times n$ matrices of quaternions. For every right eigenvalue λ of A there exists a quaternion α such that $\alpha^{-1}\lambda\alpha$ (which is also a right eigenvalue) is contained in the union of the Geršgorin balls $\{q \in \mathbb{H} : |q - a_{ii}| \leqslant R_i(A)\}$, i.e.,*

$$\{z^{-1}\lambda z : 0 \neq z \in \mathbb{H}\} \cap \bigcup_{i=1}^{n} \{q \in \mathbb{H} : |q - a_{ii}| \leqslant R_i(A)\} \neq \emptyset.$$

In particular, when λ is real, it is contained in a Gersgŏrin ball.

Proof. Let $Ax = x\lambda$, where $x = [x_1, \ldots, x_n]^{\mathrm{t}}$ is a nonzero quaternionic column vector. Let x_t be such that $|x_t| \geqslant |x_i|$ for all i. Then $|x_t| > 0$.

Using the tth row of A to multiply x, the equation system $Ax = x\lambda$ implies

$$\sum_{j=1}^{n} a_{tj}x_j = x_t\lambda.$$

Since $x_t \neq 0$, let β be such that $x_t\lambda = \beta x_t$. Then β is similar to λ and thus

$$(\beta - a_{tt})x_t = \sum_{j=1, j\neq t}^{n} a_{tj}x_j,$$

which yields, by the triangle inequality,

$$|\beta - a_{tt}||x_t| = \left|\sum_{j=1, j\neq t}^{n} a_{tj}x_j\right| \leqslant \sum_{j=1,\ j\neq t}^{n} |a_{tj}||x_j| \leqslant |x_t| \sum_{j=1,\ j\neq t}^{n} |a_{tj}| = |x_t|R_t(A).$$

It follows that $|\beta - a_{tt}| \leqslant R_t(A)$ and we conclude that β lies in the union of all Geršgorin balls. Notice that β is also a right eigenvalue of A. $\square$

While the first part of the Geršgorin theorem may be found in many texts, see, for instance, [22, p. 4], the second part that a connected region of k discs contains k eigenvalues can be seen in [11, p. 344] or [4, p. 244]. The analog for quaternionic matrices is as follows. Its proof, using the continuity property of the eigenvalues, is rather similar to that of our Theorem 5 and is parallel to that in [11, p. 344]. We thus omit it.

Theorem 8. *If k Geršgorin balls are connected and disjoint with other Geršgorin balls, then the connected region contains at least k left (as well as right) eigenvalues of the matrix. (Some eigenvalues may be counted more than once.)*

3. Examples

Huang and So make great effort in computing the left eigenvalues of 2×2 and 3×3 quaternionic matrices [13,20]. However there is no systematic approach available yet for matrices of higher dimensions. The computation of the left spectrum of a 2×2 matrix is reduced to solving a quaternionic quadratic equation. Note that over the quaternions, $\lambda^2 = 1$ has only two solutions ± 1, while $\lambda^2 = -1$ has infinitely many solutions (S^2-sphere), including $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. The cardinality of the left spectrum has a lot to do with the number of solutions of a quaternionic equation, which is impossible or very difficult to deal with in general. In mathematics examples are sometimes rather powerful; they may disprove a claim or reveal insights of a theorem. This section aims to present through examples the differences between the complex matrices and the quaternionic matrices focusing mainly on the left and right eigenvalue problems. One will see that some true statements of real or complex matrices that we take for granted may no longer hold for quaternionic matrices, and some results for right eigenvalues may be invalid for left eigenvalues, and vice versa.

Example 3.1. A quaternionic matrix may have infinitely many *left* eigenvalues as well as infinitely many *right* eigenvalues. Take the 2×2 real matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then

$$\sigma_{\mathrm{l}}(A) = \sigma_{\mathrm{r}}(A) = \left\{\lambda \in \mathbb{H} : \lambda^2 + 1 = 0\right\} = \{q\mathbf{i}q^{-1} : 0 \neq q \in \mathbb{H}\}.$$

That is, the left and right spectra $\sigma_{\mathrm{l}}(A)$ and $\sigma_{\mathrm{r}}(A)$ both have infinitely many elements. In particular, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are left and also right eigenvalues of A.

Example 3.2 [26, *Example* 5.2]. The left and right spectra of a quaternionic matrix may have no elements in common; i.e., $\sigma_l(A) \cap \sigma_r(A) = \emptyset$. Take

$$A = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{j} & 0 \end{bmatrix}.$$

Then

$$\sigma_l(A) = \left\{ \pm \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \right\} \quad \text{and} \quad \sigma_r(A) = \{\lambda \in \mathbb{H} : \lambda^4 + 1 = 0\}.$$

It is easy to check that no left eigenvalue is a right eigenvalue. Note that A has two left eigenvalues and infinitely many right eigenvalues. Furthermore, the complex number $\frac{1}{\sqrt{2}}(1 + \mathbf{i})$ is a right eigenvalue.

Example 3.3 [26, *Example* 5.3]. A quaternionic matrix may have finite right eigenvalues but infinitely many left eigenvalues. Also left and right eigenvalues may be different even for Hermitian matrices. Take

$$A = \begin{bmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{bmatrix}.$$

Then $A = A^*$, i.e., A is Hermitian. Obviously $\sigma_r(A) = \{1, -1\}$. We also have

$$\sigma_l(A) = \{\lambda : (\mathbf{i}\lambda)^2 + 1 = 0\} = \{\lambda : \lambda = \alpha + \beta\mathbf{j} + \gamma\mathbf{k}, \alpha^2 + \beta^2 + \gamma^2 = 1\}.$$

There are two right eigenvalues but infinitely many left eigenvalues. Moreover, one may check that $1, -1, \mathbf{j}$, and $\mathbf{k}$ are left eigenvalues. Thus $\sigma_r(A) \subset \sigma_l(A)$. Note that, unlike the right eigenvalue case, $\mathbf{j}$ is a left eigenvalue of A, but the quaternion $\mathbf{i}$ which is similar to $\mathbf{j}$ is not a left eigenvalue of A. Furthermore, similar quaternionic matrices may not have the same set of left eigenvalues:

$$B = P^{-1}AP, \quad \text{where } B = \begin{bmatrix} 0 & \mathbf{k} \\ -\mathbf{k} & 0 \end{bmatrix}, \; A = \begin{bmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{bmatrix}, \; P = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{j} \end{bmatrix}.$$

Note that $\mathbf{i}$ is a left eigenvalue of B (with a corresponding eigenvector $[1, -\mathbf{j}]^t$), but $\mathbf{i}$ is not a left eigenvalue of A.

Example 3.4. If $\lambda \in \mathbb{H}$ is a left eigenvalue of a quaternionic matrix A, then $k\lambda$ is a left eigenvalue of the matrix kA for any $k \in \mathbb{H}$, since $Ax = \lambda x$ implies $(kA)x = k(Ax) = k(\lambda x) = (k\lambda)x$. This is not true for right eigenvalues. Let

$$A = \begin{bmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & \mathbf{i} \end{bmatrix}.$$

Then $\lambda = 1$ is a right eigenvalue of A, but $\mathbf{j}\lambda = \mathbf{j}$ is not a right eigenvalue of $\mathbf{j}A$. Besides, as is known, unless all of the eigenvalues are real, an $n \times n$ quaternionic matrix always has infinitely many right eigenvalues (up to n equivalence classes). The matrix B shows that a square quaternionic matrix may have finitely many, not-all-real, nonsimilar left eigenvalues. The left eigenvalues of B are 1 and $\mathbf{i}$ (which are not similar), whereas the right eigenvalues of B are 1 and all the quaternions similar to $\mathbf{i}$, namely $q\mathbf{i}q^{-1}$, $0 \neq q \in \mathbb{H}$.

Example 3.5. A quaternion λ is a left eigenvalue of a quaternionic matrix A if and only if $\lambda I - A$ is singular. This is not the case for right eigenvalues. Let λ be a right eigenvalue of A. The matrix $\lambda I - A$ is not necessarily singular. Take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{bmatrix}.$$

Then $-\mathbf{i}$ is a right eigenvalue of A, but $\lambda I - A = \begin{bmatrix} -\mathbf{i}-1 & 0 \\ 0 & -2\mathbf{i} \end{bmatrix}$ is nonsingular.

Example 3.6 [26, *Example* 7.2]. (Unitarily) Similar matrices may have different traces. In general matrix similarity is meaningless for left eigenvalues. Let

$$A = \begin{bmatrix} \mathbf{i} & \mathbf{i} \\ 0 & -\mathbf{i} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \mathbf{i} - \frac{1}{2}\mathbf{k} & \frac{1}{2}\mathbf{i} \\ -\frac{1}{2}\mathbf{i} & -\mathbf{i} - \frac{1}{2}\mathbf{k} \end{bmatrix} = U^*AU,$$

where $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -\mathbf{j} \\ -\mathbf{j} & 1 \end{bmatrix}$. Then U is unitary, but $\mathrm{tr}A = 0$ while $\mathrm{tr}B = -\mathbf{k}$.

Example 3.7 [26, *Example* 7.4]. Unlike the case for complex matrices, a matrix with distinct right eigenvalues may not be diagonalizable. Let

$$A = \begin{bmatrix} \mathbf{i} & 1 \\ 0 & \mathbf{j} \end{bmatrix}.$$

Then $\mathbf{i}$ and $\mathbf{j}$ are right eigenvalues of A with eigenvectors $[1, 0]^{\mathrm{t}}$ and $[\mathbf{i}+\mathbf{j}, 0]^{\mathrm{t}}$, respectively, which are (left and right) linearly dependent. There does not exist a nonsingular matrix P such that $P^{-1}AP$ is diagonal. We must point out that $\mathbf{i}$ and $\mathbf{j}$ are in the same equivalence class, since $\mathbf{j} = q^{-1}\mathbf{i}q$ for any $q = x_0 + x_1\mathbf{i} + x_1\mathbf{j} - x_0\mathbf{k}$, where x_0 and x_1 are real numbers such that $x_0x_1 \neq 0$. It is known that if A is an n-square quaternionic matrix and all the n equivalence classes of the right eigenvalues are distinct, then A is diagonalizable [26].

Example 3.8 [12]. A matrix having distinct left eigenvalues may not be diagonalizable. Let

$$A = \begin{bmatrix} -\mathbf{i}-\mathbf{j} & 1-2\mathbf{k} \\ 1 & -\mathbf{i}+\mathbf{j} \end{bmatrix} = \begin{bmatrix} 1 & -\mathbf{j} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -\mathbf{i} & 0 \\ 1 & -\mathbf{i} \end{bmatrix}\begin{bmatrix} 1 & -\mathbf{j} \\ 0 & 1 \end{bmatrix}^{-1}.$$

Then A has two and only two left eigenvalues

$$l_1 = p + (p^2-1)\mathbf{i} - p^3\mathbf{k} \quad \text{and} \quad l_2 = -p + (p^2-1)\mathbf{i} + p^3\mathbf{k},$$

where

$$p = \sqrt{\frac{\sqrt{5}-1}{2}} = 0.786\ldots$$

One checks by calculation that A is not diagonalizable. Note that this example also shows that similar matrices may have different left eigenvalues. Moreover A has a duplicate right eigenvalue $-\mathbf{i}$. Thus $\sigma_{\mathrm{r}}(A) = \{q^{-1}\mathbf{i}q : 0 \neq q \in \mathbb{H}\}$.

Example 3.9. For $A \in \mathbb{F}^{n\times n}$, the numerical range or field of values of A over $\mathbb{F}$ is $W_{\mathbb{F}}(A) = \{x^*Ax : x \in \mathbb{F}^n, \|x\| = 1\}$. A celebrated theorem due to Toplitz–Hausdorff asserts that the (complex) numerical range of a complex matrix is a convex subset of $\mathbb{C}$. This is not true for the quaternionic case. Take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{bmatrix}.$$

Then $\mathbf{i}$ and $-\mathbf{i}$ are right (left) eigenvalues of A. But the convex combination $\frac{1}{2}\mathbf{i} + \frac{1}{2}(-\mathbf{i}) = 0$ is not in $W_{\mathbb{H}}(A) = \{x^*Ax : x \in \mathbb{H}^n, \|x\| = 1\}$, since x^*Ax is never zero unless x is zero. Note that

$W_{\mathbb{C}}(A)$ is a convex subset of $\mathbb{C}$ when A is complex. In addition, like the complex case, every right eigenvalue falls in the numerical range; that is, if $Ax = x\lambda$, assuming $\|x\| = 1$, then $\lambda \in W_{\mathbb{H}}(A)$ since $x^*Ax = x^*x\lambda = \lambda$. However, left eigenvalues are not necessarily contained in the numerical range. Note that $\mathbf{j}$ is a left eigenvalue of $B = \begin{bmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{bmatrix}$ (see Example 3.3), but $\mathbf{j} \notin W_{\mathbb{H}}(B)$, since B is Hermitian and $x^*Bx \in \mathbb{R}$ for any x.

Example 3.10 [6, *p.* 374]. In the complex case, it is known that a matrix is nilpotent if and only if all the eigenvalues are zero. For quaternionic matrices, left eigenvalues of a nilpotent quaternionic matrix are not necessarily zero. Let

$$A = \begin{bmatrix} \mathbf{i} & \mathbf{j} \\ -\mathbf{j} & \mathbf{i} \end{bmatrix}.$$

Then $A^2 = 0$; that is, A is nilpotent. However

$$\begin{bmatrix} \mathbf{i} & \mathbf{j} \\ -\mathbf{j} & \mathbf{i} \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{i} \end{bmatrix} = (\mathbf{i} - \mathbf{k}) \begin{bmatrix} 1 \\ \mathbf{i} \end{bmatrix}.$$

Thus $\mathbf{i} - \mathbf{k}$ is a left eigenvalue of A. Note that zero is the only right eigenvalue of A since A is nilpotent, i.e., $\sigma_{\mathrm{r}}(A) = \{0\}$.

Example 3.11 [6, *p.* 374]. A quaternionic matrix with only zero left eigenvalues may not be nilpotent. Let

$$A = \begin{bmatrix} \mathbf{j} & \mathbf{i} \\ \mathbf{i} & -\mathbf{j} \end{bmatrix}.$$

If λ is a left eigenvalue of A, then there exist $x, y \in \mathbb{H}$ such that

$$\begin{bmatrix} \mathbf{j} & \mathbf{i} \\ \mathbf{i} & -\mathbf{j} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}.$$

From this we derive $\mathbf{k}\lambda x = -\lambda y$. If $\lambda \neq 0$, then $y = -\lambda^{-1}\mathbf{k}\lambda x$, we further get $\mathbf{k}\lambda^{-1} + \lambda^{-1}\mathbf{k} = \mathbf{i}$. This is impossible since the coefficient of $\mathbf{i}$ on the left hand side is always 0. Therefore A has only zero left eigenvalues, i.e., $\sigma_{\mathrm{l}}(A) = \{0\}$.

Upon computation, we have

$$A^2 = -2 \begin{bmatrix} 1 & \mathbf{k} \\ -\mathbf{k} & 1 \end{bmatrix}, \quad A^4 = 8 \begin{bmatrix} 1 & \mathbf{k} \\ -\mathbf{k} & 1 \end{bmatrix}, \ldots$$

It turns out that A^{2^m}, where m is a positive integer, is always a nonzero multiple of the matrix $\begin{bmatrix} 1 & \mathbf{k} \\ -\mathbf{k} & 1 \end{bmatrix}$. Thus $A^p \neq 0$ for any positive integer p; that is, A is not nilpotent. Alternatively, this can also be seen by finding the right eigenvalues of A. The right eigenvalues of A on the closed upper half complex plane are 0 and $2\mathbf{i}$. It is interesting to notice that the following eigenvectors all belonging to the left (also right) eigenvalue 0 are left linearly independent:

$$u = \begin{bmatrix} -\mathbf{k} \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} -\mathbf{j} \\ \mathbf{i} \end{bmatrix}, \quad w = \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \end{bmatrix}.$$

Remark. This example shows that the left and right spectral radii may be different. It also reveals difficulty of defining the multiplicity of a left eigenvalue.

Example 3.12. Let Σ be the collection of all 2×2 complex matrices of the form $\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}$, α, $\beta \in \mathbb{C}$. It is known [27] that for any $A \in \Sigma$ there exists a nonsingular matrix $P \in \Sigma$ such that $P^{-1}AP \in \Sigma$ is a diagonal matrix with the right eigenvalues λ and $\bar{\lambda}$ of A on the main diagonal, and such a $P \in \Sigma$ is not unique. However in general there are also invertible 2×2 matrices Q that are not in Σ but $Q^{-1}AQ$ is a diagonal matrix in Σ. Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad P = \frac{1}{\sqrt{2}|a|}\begin{bmatrix} a & \bar{a}\mathbf{i} \\ a\mathbf{i} & \bar{a} \end{bmatrix}, \quad 0 \neq a \in \mathbb{C}, \qquad Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ \mathbf{i} & -\mathbf{i} \end{bmatrix}.$$

Then

$$P^*AP = Q^*AQ = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix},$$

where $\lambda = \mathbf{i}$. Note that A and P are in Σ but Q is not; P, Q are unitary.

Example 3.13 [26, *Example* 7.3]. A matrix A is invertible, but its transpose A^{t} maybe not. So for any invertible matrices P and Q, $PAQ \neq A^{\mathrm{t}}$. Let

$$A = \begin{bmatrix} 1 & \mathbf{i} \\ \mathbf{j} & \mathbf{k} \end{bmatrix}. \quad \text{Then } A^{\mathrm{t}} = \begin{bmatrix} 1 & \mathbf{j} \\ \mathbf{i} & \mathbf{k} \end{bmatrix}.$$

Note that $A^*A = AA^* = 2I_2$. Thus A is invertible and normal. But its transpose A^{t} is singular since $\operatorname{rank}(A^{\mathrm{t}}) = 1$. In fact zero is an eigenvalue of A^{t}. The other right eigenvalue of A^{t} in the upper half complex plane is $1 + \mathbf{i}$. Note that A^{t} is not normal. The eigenvectors corresponding to the right eigenvalues 0 and $1 + \mathbf{i}$ are respectively

$$v_0 = \begin{bmatrix} x \\ \mathbf{j}x \end{bmatrix}, \quad 0 \neq x \in \mathbb{H},$$

and

$$v_1 = \begin{bmatrix} y \\ -\mathbf{j}y\mathbf{i} \end{bmatrix}, \quad 0 \neq y = y_0 + y_1\mathbf{i} + y_0\mathbf{j} - y_1\mathbf{k} \in \mathbb{H}.$$

The vectors v_0 and v_1 are never orthogonal and never linearly dependent. Since A is normal and A^{t} is not, A is unitarily diagonalizable but A^{t} is not. That is, there exists a unitary quaternionic matrix U such that U^*AU is diagonal, however $V^*A^{\mathrm{t}}V$ is never diagonal for any unitary matrix V, though A^{t} is diagonalizable; namely $P^{-1}A^{\mathrm{t}}P$ is diagonal for some invertible matrix P.

This example also shows, unlike the complex case [24], that the transpose A^{t} of a quaternionic square matrix A is not necessarily congruent to A.

Example 3.14. A and its transpose A^{t} may have different left eigenvalues. Take

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \mathbf{i} \\ \mathbf{j} & 0 & 0 \end{bmatrix}.$$

Then $\mathbf{k}$ is a left eigenvalue of A, but not that of A^{t}. Note that $\sigma_{\mathrm{r}}(A) = \sigma_{\mathrm{r}}(A^{\mathrm{t}})$.

In the complex case, since A and its transpose A^{t} have the same eigenvalues, one may use the Geršgorin discs of both A and A^{t}, whichever give better estimates, to locate the eigenvalues of

A. The above example shows that this idea no longer works for the quaternionic matrices. Table 1 summarizes some properties of the two sorts of eigenvalues.

Table 1
Left and right eigenvalue comparison

$A \in \mathbb{H}^{n\times n}$	Left eigenvalues λ	Right eigenvalues λ
Existence	Yes	Yes
How-many	Unknown	n equi. classes $[\lambda]$
Structure	Known for $n = 2$	use χ_A
Translation	$\sigma_l(pI + qA) = \{p + q\lambda\}$	No
Similar matrices	Maybe different	Same
Singularity	$A - \lambda I$ singular	$A - \lambda I$ no singular
Quat. similarity	$q^{-1}\lambda q$: no	$q^{-1}\lambda q$: yes
Unitary similarity	No in general Yes under permutation	$\sigma_r(U^*AU) = \sigma_r(A)$
Upper (lower) tria.	$\sigma_l = \{a_{11}, \ldots, a_{nn}\}$	$\sigma_r = \bigcup_{i=1}^{n}[a_{ii}]$
Diagonalizability	No	Yes if all cla. dist.
A is real	$\sigma_l = \sigma_r$	
λ is real	$\lambda \in \sigma_l \cap \sigma_r$	
A, λ both complex	$\lambda \in \sigma_l \cap \sigma_r$	
Numerical range	λ maybe not in $W_{\mathbb{H}}(A)$	$\lambda \in W_{\mathbb{H}}(A)$
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\sigma_l = \{\lambda : \lambda^2 = -1\}$	$\sigma_r = \sigma_l$
$\begin{bmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{bmatrix} = A^*$	$\lambda = \alpha + \beta\mathbf{j} + \gamma\mathbf{k}$, $\alpha^2 + \beta^2 + \gamma^2 = 1$; left eig. $\mathbf{j} \notin W_{\mathbb{H}}(A)$	$\sigma_r = \{1, -1\}$
$\begin{bmatrix} 2 & \mathbf{i} \\ -\mathbf{i} & 2 \end{bmatrix} > 0$	$\lambda = 2 + \beta + \delta\mathbf{j} + \gamma\mathbf{k}$, $\alpha^2 + \beta^2 + \gamma^2 = 1$	$\sigma_r = \{1, 3\}$
$\begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{j} & 0 \end{bmatrix}$	$\lambda = \pm\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$ No left eigen. is right eigen.	$\lambda^4 = -1$ $\sigma_l \cap \sigma_r = \emptyset$
$\begin{bmatrix} -\mathbf{i} - \mathbf{j} & 1 - 2\mathbf{k} \\ 1 & -\mathbf{i} + \mathbf{j} \end{bmatrix}$	Two distinct left eigen. but not diagonalizable	
$\begin{bmatrix} \mathbf{i} & \mathbf{j} \\ -\mathbf{j} & \mathbf{i} \end{bmatrix}$	$\mathbf{i} - \mathbf{k} \in \sigma_l$, $\rho_l > 0$	$A^2 = 0$, $\rho_r = 0$
$\begin{bmatrix} \mathbf{j} & \mathbf{i} \\ \mathbf{i} & -\mathbf{j} \end{bmatrix}$	$\sigma_l = \{0\}$, $\rho_l = 0$	$A^p \neq 0$, $\rho_r = 2$
$\begin{bmatrix} 0 & 1 + \mathbf{i} \\ 1 - \mathbf{i} & 0 \end{bmatrix}$	$\sigma_l(A) = \sigma_l(A^t) = \{\pm\sqrt{2}\}$	$\sigma_r(A) = \sigma_r(A^t) = \{\pm\sqrt{2}\}$
$\begin{bmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}$	$\mathbf{j} \in \sigma_l(A)$	$\mathbf{j} \notin \sigma_r(A^t)$
$\begin{bmatrix} 1 & \mathbf{i} \\ \mathbf{j} & \mathbf{k} \end{bmatrix}$	$PAQ \neq A^t$, P, Q inv.	A, A^t not congr.
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \mathbf{i} \\ \mathbf{j} & 0 & 0 \end{bmatrix}$	$\sigma_l(A) \neq \sigma_l(A^t)$, $\mathbf{k} \in \sigma_l(A), \mathbf{k} \notin \sigma_l(A^t)$	$\sigma_r(A) = \{\lambda :\ \lambda^3 \in [\mathbf{k}]\}$ $\mathbf{k} \in \sigma_r(A^t)$

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